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Toward a turbulence constitutive relation for rotating flows

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ABSTRACT

In rapidly rotating turbulent flows the largest scales of the motion are in approximate geostrophic balance. Single-point turbulence closures, in general, cannot attain a geostrophic balance. This article addresses and resolves the possibility of constitutive relation procedures for single-point second order closures for a specific class of rotating or stratified flows. Physical situations in which the geostrophic balance is attained are described. Closely related issues of frame-indifference, horizontal nondivergence, Taylor-Proudman theorem and two-dimensionality are, in the context of both the instantaneous and averaged equations, discussed. It is shown, in the absence of vortex stretching along the axis of rotation, that turbulence is frame-indifferent. A derivation and discussion of a geostrophic constraint which the prognostic equations for second-order statistics must satisfy for turbulence approaching a frame-indifferent limit is given. These flow situations, which include rotating *and* nonrotating stratified flows, are slowly evolving flows in which the constitutive relation procedures are useful. A nonlinear non-constant coefficient representation for the rapid-pressure strain covariance appearing in the Reynolds stress and heat flux equations consistent with the geostrophic balance is described. The rapid-pressure strain model coefficients are not constants determined by numerical optimization but are functions of the state of the turbulence as parameterized by the Reynolds stresses and the turbulent heat fluxes. The functions are valid for all states of the turbulence attaining their limiting values only when a limit state is achieved. These issues are relevant to strongly vortical flows as well as flows such as the planetary boundary layers, in which there is a transition from a three-dimensional shear driven turbulence to a geostrophic or horizontal turbulence.

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1. Introduction

Many geophysical and environmental flows exhibit regions of the flow which are in an approximate geostrophic balance - a balance of Coriolis and pressure forces, $2\varepsilon_{ikp}u_p\Omega_p \simeq -p_{,i}$. At present no turbulence closure for the statistics of the fluctuating field is consistent with the geostrophic balance. Single-point turbulence models are, in general, subject to *ad hoc* corrections when applied to rotating flows. This approach does not make use of the mathematical requirements that the dependent variables and their evolution equations satisfy. The difficulty with the equations, as currently modeled, can be seen when the equations are transformed to a rotating coordinate system - a geostrophic balance is not attainable in the small Rossby number limit. Closely related to this is the requirement, associated with the Taylor-Proudman theorem, that as the flow reaches a two-dimensional state the turbulence be frame-indifferent. Frame-indifference is required by the Navier-Stokes equations, Hide (1977), Speziale (1981, 1985), Ristorcelli, Lumley and Abid (1995). The problem with the current models for the second-order equations results from the inability of the rapid-pressure strain covariance model to reflect the physics associated with the reduction of vortex stretching. These issues are related to the fact that if there is no vortex stretching along the axis of rotation then rotation does not play a dynamical role in the flow's evolution. This is a form of realizability: a turbulence that is horizontally divergence free is frame-indifferent and the exact and, therefore, the modeled equations must be frame-invariant. The terms frame-indifferent and frame-invariant refer to the fact that the rate of rotation of the frame of reference does not appear in the prognostic equations.

The title to this article is after Lumley's (1970) article with a similar title. Lumley's (1970) article discussed the possibility of the use of the invariant basis methods of rational mechanics to create constitutive relations for $< u_i u_j >$. Here the fact that, in more complex flows with rotation or buoyancy, one needs to carry evolution equations for $< u_i u_j >, < u_i \theta >$, is acknowledged. Nonetheless these are slowly evolving flows and are therefore suitable candidates for the invariant basis methods. Lumley's (1970) article also discusses the principle of material-frame-indifference; a modelling principle used with success in continuum mechanics. Lumley (1970) provides a very clear physical picture for why such a principle is *not* appropriate for positing turbulence constitutive relations. This article is in concordance with that viewpoint: the principle of material-frame-indifference is *not* a valid theoretical principle for a *general three-dimensional* turbulent field. It is, however, an applicable principle *in the limit* of a horizontally nondivergent field for which the equations are frame-invariant. Research in rotating turbulence typically addresses issues relating to the reduction of the cascade rate. Here the focus will be on the Reynolds stresses and the form their modeled evolution equation takes when the stretching of vorticity along the axis of rotation

vanishes. This sort of flow structure takes place in strongly vortical flows of aerodynamic interest and also in many geophysical situations.

This article discusses in depth these and attendant analytical issues. A substantial portion of the time is spent describing flow situations in which rotation or stratification plays a role before indicating how these issues manifest themselves in the instantaneous quantities and their statistics. The intention is to illustrate the physical effects of rotation and stratification and to show how such phenomena lead to flows whose evolution equations are frame-invariant. The point made is that, for flows affected by rotation or stratification, a two-dimensional or horizontal turbulence is an equilibrium situation and the constitutive relation procedures are viable. Mathematical details regarding Taylor-Proudman, frame-indifference and horizontal divergencelessness are then discussed. These issues are discussed in the context of three flows, rotating, baroclinic and stably stratified. In these flows the primary component of the fluctuating vorticity is aligned with the axis of rotation. It is then shown how these issues are dealt with in the context of second-order moment closures; a constitutive relation for the pressure-strain covariance ensuring that the proper frame-invariance of the modeled second-order moments equations is produced.

The potential importance of frame-indifference for turbulence modelling was put forward, in the context of a rapidly rotating mechanical turbulence, by Speziale (1981, 1985). In this article this idea has been substantially generalized to a wider class of flows, both rotating *and* nonrotating. The proofs of frame-invariance used here are also different than that of Speziale (1981, 1985). The present proofs deal directly with the vorticity and as such allow a very clear physical intepretation in terms of vortex stretching. This allows a comprehensive assessment of the class of flows for which such a principle might be useful. This article also indicates how these frame-indifference issues are used to produce a properly frame invariant set of modeled equations.

In short we (1) describe the physical effects of rotation and stratification in a number of stationary flows, (2) show the mathematical consequences of rapid rotation or stable stratification, (3) indicate how the geostrophic balance manifests itself in the second-moment equations and (4) discuss the possibility of and posit a turbulence constitutive relationship consistent with the geostrophic balance. It is made clear that such a single-point constitutive procedure is limited to classes of turbulence for which the anisotropy of the Reynolds stresses and heat fluxes is a suitable measure of the anisotropy of the turbulence.

2. Frame invariance in rotating and stratified flows

Several examples of flows in which stratification and rotation lead to frame-indifferent flows are

described. This will motivate and clarify subsequent more mathematical considerations. To provide physical insight into these effects three simple cases - the rotating tank, the rotating differentially heated annulus, and the stratified grid turbulence - are treated in detail. In all three of these case there are regions in which a horizontally divergence-free stationary turbulence exists.

A few points on nomenclature are appropriate: A geostrophic turbulence is one whose horizontal divergence is small because its Rossby number is small. This latter qualification distinguishes it from a general flow with vanishing horizontal divergence for arbitrary Rossby number. Reynolds (1989), has coined the word componential to better distinguish the several different classes of flows for which the phrase two-dimensional is casually used. Componential refers to the number of components the velocity vector has; dimensional refers to the number of independent variables of which it is a function. Thus a 2D-2C field is $u_i(x,y), i = 1,2$ while a 2D-3C is one in which $u_i(x,y), i = 1,2,3$. See also Kassinos and Reynolds (1994). Lesieur (1991) has also found the need to differentiate these different types of fields. For Lesieur (1991) a two-dimensional turbulence, as seen in the Taylor-Proudman situations has $u_i(x,y), i = 1,2,3$ while a turbulence with $u_i(x,y,z), i = 1,2$, as arises in stably stratified situations, is called a horizontal turbulence. Both of these fields in the limit of small Rossby or Froude number are frame-indifferent.

The rotating tank: Figure 1 shows a schematic of the three different regimes seen in rotating grid generated turbulence studied by Hopfinger, Browand and Gagne (1982). It is an inhomogeneous turbulence with no mean flow; an isotropic turbulence is generated at an oscillating grid placed at the bottom of the rotating tank. Greenspan (1968), has tentatively summarized the major effects of rotation as being the tendency of the flow to become two-dimensionalized in planes perpendicular to the axis of rotation, the occurrence of inertial wave for frequencies, $\omega < 2\Omega$ and the rapid spin up of fluid elements or creation of intense vortices. Hopfinger, Browand and Gagne's (1982) rotating tank, the first example below, is very simple and elegant experiment exhibiting all three of these aspects. This flow is a graphic demonstration of the effects of rotation as a function of the local turbulent Rossby number, $Ro_t = \tilde{u}/2\Omega\ell$, which changes in the inhomogeneous (axial) direction. The axial coordinate is a proxy for the Rossby number which decreases with from distance from the grid as, near the grid, $\tilde{u} \sim z^{-1}$ and $\ell \sim z$ and $Ro_t \sim z^{-2}$.

The grid is oscillated at a frequency much larger than $f > 2\Omega$. Near the grid one obtains an isotropic three-dimensional turbulence unaffected by rotation as seen in inertial systems. This is a region of large Ro_t; as long as the turbulence frequency $\tilde{u}/\ell > 2\Omega$ the usual three-dimensional turbulence exists.

Further from the oscillating grid, $Ro_t < 0.4$, there is a transitional zone in which the largest scales

of the motion are two-dimensional over which are superposed smaller scale ageostrophic motions. In this region the energy decay is substantially diminished; this is due to a reduction of vortex stretching and thus the reduction of the cascade from the energy containing two-dimensional scales. The smaller ageostrophic motions continue to lose energy through the cascade and dissipation. The fluctuating field is composed of inertial waves and turbulence.

Further past the transitional zone there is a third region in which the turbulence, now almost purely two componential is independent of the axial coordinate, two-dimensional - "2D-2C". The transition to this flow takes place abruptly at $Ro_t = 0.2$. The flow is composed of thin coherent columnar vortices whose lifespans are very large compared to the rotation period, $(2\Omega)^{-1}$ and the eddy turnover time, $\ell/\tilde{u}$. The flow is consistent with the Taylor-Proudman theorem:

$$\Omega_j u_{i,j} \to 0. \tag{1}$$

In this region of the flow both energy and enstrophy conserved quantities and as a consequence the transfer of energy from a larger to a smaller scale of the flow is accompanied by transfer from smaller to larger wavenumber. The "inverse cascade" imposes a powerful constraint on nonlinear interactions between different scales of the motion. It is accompanied, as is well known, by vortex merging phenomenon. The vorticity of the vortices can be two orders of magnitude larger than the background rotation.

The differentially heated rotating annulus: Both buoyancy and rotation are important in this flow situation, Figure 2. This experimental set up simulates the transport of heat in the atmosphere (*modulo* the beta effect) from the warm equatorial regions to the poles: the heated outer cylinder representing the equatorial regions, the cool inner cylinder the polar regions, the azimuthal direction represents the zonal (east-west) flows. A horizontal temperature gradient is impressed across a rotating annulus. Under the action of strictly gravitation torques the fluid elements undergo an overturning motion and the differential heating produces a stably stratified meridional circulation. The mean vorticity is azimuthal. At zero and low rotation rates the flow is an "ageostrophic" thermal turbulence (provided the Grashof number is high enough). As the annulus is rotated more rapidly several things occur: Coriolis forces inhibit the meridional flow, vertical component of the velocity is reduced, and an azimuthal component to the mean velocity is induced. The Coriolis force that inhibits the overturning motion in the meridional planes and promotes a different kind of convection called sloping convection, Hide and Mason (1975): fluid elements have trajectories with very small angles to the horizontal. This is a manifestation of baroclinic waves. The waves transfer heat and momentum perpendicular to the shear.

Some of the features of this sequence of transitions can be understood from the thermal wind

balance. The momentum equations, in this limit, reflect the geostrophic and hydrostatic balances,

$$2\Omega_3 U_2 \simeq p_{,1}\,, \qquad 2\Omega_3 U_1 \simeq p_{,2}\,, \qquad g\beta T \simeq -p_{,3}\,. \tag{2}$$

Here the coordinates $[x_1, x_2, x_3]$ are the cartesian equivalents to the cylindrical system $[r, \theta, z]$. The comma indicates differentiation with respect to the subscripted variable. The basic state represents a balance between buoyancy, Coriolis and pressure forces in the interior of the fluid. The equations for the azimuthal and vertical components of the vorticity become

$$2\Omega_3 U_{2,3} \simeq g\beta T_{,2}\,, \qquad 2\Omega_3 U_{3,3} \simeq 0. \tag{3}$$

If the basic state of the temperature field is assumed to be of the form $T(x_1, x_3) = x_3 - x_1 tan(\theta)$ the balance requires an azimuthal shear, $U_2 = \frac{g\beta T_{,1}}{2\Omega} z$. This is the basic state that is used in the Eady model for the analysis of the baroclinic instability, Hide and Mason (1976). It is valid when the Ekman and Rossby numbers are small. The baroclinic instability is the instability of the vertically sheared zonal current, $U_2 = \frac{g\beta T_{,1}}{2\Omega} z$.

The different flow states as a function of two external flow parameters reflecting the size of the impressed horizontal temperature gradient and the speed of the rotation is of interest. For a given Prandtl number the flow can be characterized by two nondimensional numbers: the thermal Rossby number, $Ro_{th} = g\beta\Delta T H/\Omega^2(R_o - R_i)^2$, and a Taylor number, $Ta = 4\Omega^2(R_o - R_i)^2/\nu^2$ where R_o and R_i are the outer and inner radii of the annulus. This sequence of transitions, associated with the geostrophic instability, are nicely illustrated in the heated rotating annulus flows of Buzyna, Pfeffer and King (1984), Fein and Pfeffer (1976) and Hide and Mason (1975). The temperature field and fluctuations were measured in these experiments. As the reason for undertaking these experiments was primarily an investigation of the transition to a geostrophic turbulence the stabilization of the ageostrophic or thermal turbulence by rotation was, from the viewpoint of the turbulence statistics, not fully explored. Furthermore the thermal turbulence (when it existed), most notably in the mercury experiments of Fein and Pfeffer (1976), has a low Reynolds number; $Re = 2500$ based on annulus width.

1). *Axisymmetric*: At large values of the Rossby number, $Ro_{th} >> 1$ (low rotation rates) the flow in the annulus is essentially buoyancy driven with a unicellular meridional circulation. The flow field is axisymmetric and the mean vorticity is azimuthal. As the rotation increases Coriolis forces reduce the azimuthal component of the mean vorticity, the vertical component of the velocity is reduced, and the mean velocity becomes zonal. This is accompanied by a reduction in heat transfer across the annulus. In addition, the stable stratification (associated with the meridional circulation driven by the horizontal heat flow) decreases, the temperature fluctuations decrease and conduction becomes a much more important component of the heat transfer.

2). *Non axisymmetric - steady baroclinic waves*: As $Ro_{th} \rightarrow 1$ the flow loses its axisymmetry: the zonal (azimuthal) velocity becomes wavelike and the primary vorticity component is vertical. There is a transition from a thermal turbulence to a turbulence field over which are superposed baroclinic waves. As the rotation increases the thermal turbulence is suppressed. There are large azimuthal waves, wavenumbers 1 and 2, travelling with a small uniform drift with respect to the rotation speed of the annulus. The waves have length scales on the order of the Rossby radius of deformation - $\lambda \sim 2(g\beta\Delta TH)^{1/2}/\Omega$. The power spectrum of the temperature has a few peaks corresponding to two characteristic wave frequencies and their harmonics. Because of the waves the heat transfer is augmented: a fluid particle gains and loses heat as it is moved back and forth by the passage of a baroclinic wave. This form of thermal convection is known as sloping convection where radial heat transfer is not by the overturning of an eddy in the meridional plane but by the passage through the fluid of an almost horizontal baroclinic wave. This is called the regular wave regime; the flow is, disregarding drift, periodic. Additional increases in rotation create additional waves with ever higher azimuthal wavenumber.

The baroclinic wave regime and ensuing instability are a result of a modification of the gravitational body forces by system rotation; what is a stably stratified temperature field in the non-rotating coordinate system becomes, at high rotation rates, an unstably stratified field due to a change in the direction in which a displaced particle oscillates. This is schematically illustrated in Figure 3. The isopycnals (isotherms) in the interior are almost horizontal, while stably stratified in the vertical.

3). *Non axisymmetric - unsteady baroclinic waves*: Further increases in the rotation rate produce, near $Ro_{th} \simeq 0.1$, the structural vacillation regime: another frequency appears and the waves undergo fluctuations in amplitude and wavenumber. With further increases in rotation rate additional frequencies appear in the temperature spectrum and background noise begins to increase. The wavelengths of the azimuthal waves continue to get smaller.

4). *Non axisymmetric - geostrophic turbulence*: When $Ro_{th} < 0.1$ the flow is spatially (in the horizontal) and temporally uncorrelated; the flow is perfectly correlated in the vertical. The temperature spectrum is broad and relatively featureless. The heat flux across the annulus undergoes a sudden decrease as the flow becomes a geostrophic turbulence; essentially composed of random unsteady baroclinic waves. With increasing rotation more baroclinic waves of higher azimuthal wavenumber appear; this is a necessary prerequisite for the transition to a baroclinic turbulence which requires an inverse cascade and cannot occur until there are enough small scales for a spectral flux to larger scales.

Stably stratified flows: The analogy between rotating and stratified fluids is known, Veronis (1970) and Hopfinger (1987); the stable stratification, as quantified by the mean density gradient, plays a role analogous to the rotation. A scaling analysis of the continuity and vorticity equations can be used to scale the relative magnitudes of the vertical and horizontal components of velocity. Linearizing around the basic stratified state the continuity equation becomes,

$$u_j \rho,_j + u_3 \frac{d\rho_0}{dz} = 0 \qquad \Rightarrow \qquad \rho \sim \frac{u_3 \ell}{u_H} \frac{d\rho_0}{dz}. \tag{4}$$

The balance in the vorticity equation produces

$$\rho_0 (u_j \omega_i,_j - \omega_j u_i,_j) = \varepsilon_{imq} g_q \rho,_m \qquad \Rightarrow \qquad \rho \sim \rho_0 \frac{u_H^2}{g\ell}. \tag{5}$$

Combining the two estimates produces

$$\frac{u_3}{u_H} \sim \rho_0 \frac{u_H^2}{g\ell} \left|\frac{d\rho_0}{dz}\right|^{-1} \sim \frac{1}{Ri}. \tag{6}$$

Ri is the Richardson number. As the stable stratification increases Ri increases and $u_3 \to 0$ and the turbulence becomes a horizontal turbulence. For arbitrary rotation the axial stretching of the vertical vorticity goes to zero: $2\Omega_3 u_{3,3} \to 0$ as $u_3 \to 0$. These scalings are consistent with the observation of the turbulence collapse for grid turbulence under stable stratification, Browand and Hopfinger (1982) and Browand, Guyomar and Yoon (1987), observed to take place when

$$\frac{\tilde{u}}{N\ell} < 0.3, \qquad N^2 = \frac{g}{\rho_0} \left|\frac{d\rho_0}{dz}\right|. \tag{7}$$

This is analogous to the collapse of the two-dimensional flow seen far from the grid in the rotating tank flows, Hopfinger, Browand and Gage (1982). The Brunt-Väisälä frequency now plays the role the Coriolos frequency, 2Ω; N^2 and 2Ω both represent upper bounds on the frequency range of gravity and inertial waves. They approximate the frequency with which a particle displaced from its equilibrium position oscillates. A similar collapse is seen in the wake of a sphere in a stably stratified environment: a fully three-dimensional turbulent wake collapses to create a two-dimensional wake, in the plane of the stratification, similar to the two-dimensional von Karman vortex street behind a cylinder. Some of these wake issues are summarized in Thorpe (1987) and Hopfinger (1987). Lin and Pao (1979) and Xu, Fernando and Boyer (1995) provide more detail.

Applications: The above flows are simple pedagogical examples useful for illustrating the physics of the effects of rotation and buoyancy which must be incorporated into single-point turbulence closures for such flows if they are to be predictive tools. These effects play a role in more complex flows.

There are the planetary boundary layers in which many of these issues are relevant. At the large scales of the motion the turbulence is two-dimensional; at the smaller scales the turbulence is three-dimensional, Wyngaard (1992). From a mesoscale point of view both of these regimes are important. For the interfaces between the atmosphere and the earth or the ocean there is a boundary layer typically a kilometer or two in height (depending on wind conditions) in which the flow makes a transition from a three-dimensional shear driven boundary layer turbulence with its attendant three-dimensional cascade mechanism to a quasi-geostrophic turbulence whose largest scales do not undergo the vortex stretching and rotation seen in three-dimensional turbulence. In the neutrally stable boundary layer rotation affects the flow within the mixed layer, Wyngaard (1992). For the stable mixed layer the size of the layer is established at the inversion cap at which the vertical component of the turbulence is extinguished by the stratification.

A similar transition takes place at the ocean–air interface and at the ocean bottom. These boundary layer while only 0.1-.5 kilometers in size represent a transition zone from three-dimensional shear or buoyancy driven turbulence to a geostrophic flow in the interior of the oceans. Computations that span these regions for either quantitative environmental calculations or to create simple parameterizations of these regions to use as boundary conditions for global circulation models will require a set of equations consistent with a geostrophic balance. This flow is complicated by the fact that the mean flow is much lower than a typical fluctuating velocity. The usual mean production mechanisms are no longer the most important and more care must be taken with the modelling of pressure and transport terms in the equations.

There are several flows of technological interest in which material-frame-indifference will potentially play a roll. Figure 4 is a sketch of a Czochralski crystal growth apparatus. The melt flow is a thermally driven flow driven by the temperature difference between the crucible and the crystal. It is modified by centrifugal forces associated with the rotation of the crystal and Coriolis forces associated with the rotation of the crucible. Some of the features seen in the baroclinic instability in the melt are seen in this flow. A summary of issues associated with this flow is given in Ristorcelli and Lumley (1992).

In vortex flows of aeronautical interest the persistence of trailing vortexes is associated with the strong rotational forces that reduce turbulence mixing. The alignment of the vortical fluctuations with the mean vorticity will also produce a frame-indifferent flow field. This situation is further exacerbated by the local effects of stable atmospheric stratification that substantially reduce their decay. A strong mean strain, as might be seen when turbulence goes through a rapid axisymmetric contraction, will align the fluctuating vorticity with the primary strain axis. Such a 2C turbulence

associated with the 1C vorticity, Figure 5, is also frame-indifferent.

3. Taylor-Proudman, frame-indifference and horizontal-divergencelessness

First it is shown that rapid rotation leads to flows whose prognostic equations are frame invariant. This is a result of the Taylor-Proudman theorem which indicates that scales of the motion with time scales slow compared to the rotational period become horizontally divergence free. It is then shown that, *for arbitrary rotation rate* and thus a much wider class of flows, any horizontally divergence free flow is frame-indifferent.

The Navier-Stokes equations, in the Bousssinesq approximation, are

$$\dot{u}_i + u_j u_{i,j} + 2\varepsilon_{ikp}\Omega_k u_p = -p_{,i} + g\beta_i\theta + \nu u_{i,jj}\,. \tag{8}$$

The pressure has been normalized by the constant density. The curl of the Navier-Stokes equations produces the vorticity equation

$$\dot{\omega}_i + u_j\omega_{i,j} = (\omega_j + 2\Omega_j)u_{i,j} + \varepsilon_{imq}g\beta_q\theta_{,m} + \nu\omega_{i,jj}\,. \tag{9}$$

where $\omega_k = \varepsilon_{kqi}u_{i,q}$. For ease of discussion it shall be assumed that the rotation and gravity are along the "3" axis. For rapidly rotating flows the Taylor-Proudman theorem, for those portions of the flow whose frequency is slow with respect to the rotation of the frame, is obtained from the vorticity equation,

$$\Omega_3 u_{i,3} \rightarrow 0 \tag{10}$$

as $\Omega_j \rightarrow \infty$. This is a statement about a flow occurring in the presence of a background vorticity of 2Ω. The motions of a fluid in geostrophic balance is the same in planes perpendicular to the axis of rotation. In such a flow the vorticity generation mechanisms of stretching along the axis of rotation $\Omega_3 s_{33}$ and rotating into the axis of rotation $\Omega_3 w_{\alpha 3}, (\alpha = 1,2)$ are reduced. This manifests in the reduction of the turbulence cascade rate. Here $s_{ij} = \frac{1}{2}[u_{i,j} + u_{j,i}]$ and $w_{ij} = \frac{1}{2}[u_{i,j} - u_{j,i}]$.

It has been shown that as $\Omega_j \rightarrow \infty$ that $\Omega_j u_{i,j} \rightarrow 0$ and the evolution equations are frame invariant. Clearly if $\Omega_j u_{i,j} \rightarrow 0$ *for arbitrary rotation rate* the flow is also frame indifferent. Situations in which $\Omega_j u_{i,j} = 0$ occur in flows with strong stable stratification (described above), low aspect ratio or MHD. For baroclinic flows the thermal wind balance of the vorticity equation is

$$2\Omega_j u_{i,j} = -\varepsilon_{imq}g\beta_q\theta_{,m}\,. \tag{11}$$

The individual components of the vorticity equation require, for $i = 1,2$, $2\Omega_j u_{i,j} = -\varepsilon_{imq}g\beta_q\theta_{,m}$, and for $i = 3$, $2\Omega_j u_{3,j} = 0$. In such flows the velocity field, to lowest order in Ro_t, is described by

a streamfunction

$$u_p = \epsilon_{pqk}\hat{\Omega}_k\psi_{,q} + w\delta_{i3} \qquad \psi_{,p} = \epsilon_{qpk}\hat{\Omega}_k u_q + \hat{\Omega}_p\hat{\Omega}_q\psi_{,q} \tag{12}$$

where $\psi = \psi\ (x, y, z)$. The carat indicates a unit vector. This is easily seen by contracting on the the geostrophic balance in the following manner: $\varepsilon_{ijq}\Omega_q\ [\ 2\varepsilon_{ikp}\Omega_k u_p = -p_{,i} + g\beta_i\theta\ ]$. Carrying through the mathematics one obtains $\psi = -p/2\Omega$. The streamfunction and pressure field are proportional and the isobars and streamlines are aligned. Thus $u \sim -p_{,y}$ and $v \sim p_{,x}$ and to lowest order $u_{,x} + v_{,y} = 0$. The hydrostatic balance, $p_{,3} = g\beta\theta$ leads to $\theta = \frac{-2\Omega}{g\beta}\psi_{,3}$. The vertical velocity is then obtained diagnostically from $\frac{D}{Dt}\theta = 0$. The vorticity is related to the streamfunction, $\psi = \psi(x, y, z)$, by

$$\omega_i = \hat{\Omega}_l\psi_{,il} - \hat{\Omega}_i\psi_{,qq}\,. \tag{13}$$

The vertical component of the vorticity is given by the horizontal Laplacian of the streamfunction $\omega_3 = -\nabla^2_H\psi$ which evolves according to

$$\dot{\omega}_3 + u_j\omega_{3,j} = (\omega_j + 2\Omega_j)u_{3,j} + \nu\omega_{3,jj}\,. \tag{14}$$

This is the prognostic equation for the streamfunction and it is closed with respect to the streamfunction. Inspection shows that it is frame-invariant if there is no axial stretching of vorticity, $\Omega_j u_{3,j} = 0$: this is a statement that the flow is horizontally divergence free: $-w_{,z} = u_{,x} + v_{,y} = 0$. As a consequence *any horizontally divergence free velocity field is also frame-indifferent.* Such a flow field, with the representation $u_p = \epsilon_{pqk}\hat{\Omega}_k\psi_{,q} + w\delta_{i3}$, includes a horizontal turbulence whose velocity components lie in planes perpendicular to the axis of rotation as well as three-dimensional velocity fields that are independent of the coordinate along the axis of rotation. It is well known that the vertical variability in these flows will in fact induce an axial stretching and that evolution of such a system is described by the conservation of potential vorticity. Our interest lies with constructing a turbulence closure consistent with physical principles: if the axial stretching (equivalently if the horizontal divergence vanishes) then the evolution equations for the turbulence must be frame invariant. Some special cases are now indicated.

Case 1: $\Omega_j u_j = 0, \psi = \psi\ (x, y)$: This is the case that Reynolds (1989) calls 2C-2D - the velocity vector has only two components and is dependent on two coordinates. In geophysical fluid dynamics, for example, the shallow water limit of the atmosphere or ocean, such a field is sometimes called two-dimensional. The field is horizontally divergence free. The Taylor-Proudman theorem applied in a bounded domain, with no flux boundary conditions, is included in this category.

Case 2: $\Omega_j u_{i,j} = 0, \psi = \psi\ (x, y)$: Such a field is horizontally divergence free. This might be called a 3C-2D meaning the flow is independent of the coordinate along the axis of rotation $\psi = \psi\ (x, y)$

and the velocity vector has three components, $u_p = \epsilon_{pqk}\Omega_k\psi_{,q} + \Omega_p u_i \delta_{i3}$. The Taylor-Proudman theorem in an unbounded domain is included in this category. Note that in the very common configuration, a bounded domain with no flux boundary conditions ($\Omega_p u_p = 0$) that the $\Omega_j u_{i,j} = 0$ case includes Case 1.

Case 3: $\Omega_j u_j = 0, \psi = \psi\,(x, y, z)$: Using Reynolds (1989) nomenclature this case would also be called 2C-3D, as $w = 0$. The streamfunction is a function of all three-coordinate directions and the field and $u_{,x} + v_{,y} = 0$.

Case 4: $\Omega_j u_{3,j} = 0, \psi = \psi\,(x, y, z)$: The streamfunction is a function of all three-coordinate directions though now $w = w\,(x, y)$. This is the most general of all four flow fields and includes cases 1, 2 and 3 as special cases and is a statement of the horizontal divergence free condition, $w_{,3} = 0$. A dependence on all three coordinate directions occurs in the case of the thermal wind balance: a horizontal temperature gradient in a rotating fluid induces vertical shear of the horizontal velocity field as can be seen from the vorticity equation, $2\Omega_3 u_{k,3} = -g\beta\theta_{,q}\varepsilon_{kq3}$. The pressure field which is related to the streamfunction, determines the density field through the hydrostatic balance of the vertical momentum equations, $\psi_{,3} = \frac{-1}{2\Omega}p_{,3} = \frac{-\beta}{2\Omega}\theta$. In the absence of such body forces the flow is not dependent on the vertical coordinate and the thermal wind balance produces $\Omega_p u_{i,p} = 0$ which may have two or three nonzero velocity components depending on initial conditions or boundary conditions.

4. Stratification, low Froude number and frame-indifference

The previous section addressed rapidly rotating flows, with and without buoyancy, in which there was an approximate geostrophic balance. It was also shown that, for arbitrary rotation rate (ageostrophic turbulence), a horizontally nondivergent field is also frame-indifferent. A turbulence in a stably stratified environment is an example of an ageostrophic turbulence that is frame-indifferent in the limit of strong stratification. The case of a stably stratified horizontal turbulence is now treated. In the strongly stably stratified case while the vertical velocity associated with turbulence does go to zero one might still expect large horizontal divergences associated with fluid elements oscillating about the equilibrium position.

A turbulent Froude number, $Fr_t = \tilde{u}/N\ell$ where $N^2 = g\beta T_{,3}$ is used to describe the properties of stably stratified turbulence. It can be understood as a ratio of length scales. The turbulence length scale, ℓ and the buoyancy length scale $\ell_b = \tilde{u}/N$. The buoyancy length scale ℓ_b can be interpreted as the largest size eddy capable of executing an overturning motion against the stratification. A three componential turbulence will in general collapse to a horizontal turbulence for small enough

Fr_t. Essentially vertical motions must perform work against the stable stratification to maintain themselves, horizontal motions do not. As the relative stratification increases energy is fed into the large scale horizontal motions and the flow collapses into striated layers sometimes described as being comprised of thin horizontal pancake vortices decorrelated in the vertical direction. The layers are independent of each other and lead to a vertical structure with different layers evolving independently (unlike the baroclinic case). The energy containing scales of the turbulence has primarily a vertical vorticity with small vertical velocities. Superposed on the turbulence will be an internal wave field with vertical velocity but no vertical vorticity. The occurrence of these vertical velocities is what distinguishes this case from the rapidly rotating flow. The wave and turbulence are, in the low Froude number limit, decoupled and the evolution equations for the turbulence are frame invariant.

The equations are written in horizontal and vertical velocity components: $u_i = v_i + w\delta_{i3}$ where $v_i = [v_1, v_2, 0]$. Continuity is therefore $u_{i,i} = v_{j,j} + w_{,3} = 0$. The momentum and energy equations, *modulo* diffusivity and viscosity, are

$$\dot{v}_i + u_j v_{i,j} + 2\varepsilon_{ikp} v_p \Omega_p + p_{,i} = 0 \tag{15}$$
$$\dot{w} + u_j w_{,j} + p_{,i} - g\beta\theta = 0 \tag{16}$$
$$\dot{\theta}_i + u_j \theta_{,j} + w\frac{dT}{dz} = 0. \tag{17}$$

A scale analysis similar to Lilley's (1983), in which the distinction between the wave fields and the turbulence is highlighted, is useful. The equations are rescaled by characteristic turbulent velocities in horizontal, u_c, and vertical directions, w_c, with length scale ℓ. *The fast time scale:* the nondimensional equations are rescaled on the wave time scale, N,

$$\dot{v}_i + p_{,i} + \frac{2\Omega}{N}\varepsilon_{ikp} v_p \hat{\Omega}_p = -Fr_t\,[u_j v_{i,j}] \tag{18}$$
$$\dot{w} + p_{,i} - g\beta\theta = -Fr_t\,[u_j w_{,j}] \tag{19}$$
$$\dot{\theta}_i + w\frac{dT}{dz} = -Fr_t\,[u_j \theta_{,j}] \tag{20}$$
$$v_{j,j} + w_{,3} = 0 \tag{21}$$

where $Ro_t = u_c/(2\Omega\ell)$ and $Fr_t = u_c/(N\ell)$. For the wave regime $w_c \sim u_c$; note that the horizontal divergence for this field is not small. In the absence of rotation (strictly for convenience of presentation) the three and two-dimensional Poisson equations for the pressure are, in the limit of small Fr_t, $\nabla^2 p = \theta_{,3}$ and $\nabla^2_H p = \dot{w}_{,3}$; the energy equation is used to produce a prognostic equation for the potential,

$$\nabla^2 p_{,tt} + \nabla^2_H p = 0, \tag{22}$$

where $\dot{\phi} = -p$. *The slow time scale:* the equations are rescaled on the advective time scale, u_c/ℓ,

$$\dot{v}_i + v_j v_{i,j} \; + Ro_t^{-1} 2\varepsilon_{ikp} v_p \hat{\Omega}_p \; + \; p_{,i} \;\; = \;\; -Fr_t^2 \, [w v_{i,3}] \tag{23}$$

$$p_{,i} \; - g\beta\theta \;\; = \;\; -Fr_t^2 \, [\dot{w} + v_j w_{,j} \; + Fr_t^2 \, [w w_{,3}]] \tag{24}$$

$$\dot{\theta}_i + v_j \theta_{,j} \; + w\frac{dT}{dz} \;\; = \;\; -Fr_t^2 \, [w_j \theta_{,3}] \tag{25}$$

$$v_{j,j} \;\; = \;\; -Fr_t^2 \, [w_{,3}]. \tag{26}$$

For the turbulence $w_c \sim Fr_t^2 u_c$. The horizontal divergence for the turbulence field, unlike the wave field, vanishes as Fr_t^2 and the vorticity is vertical. Such a field is fully described by a streamfunction, $\psi = \psi(x, y, z)$ whose evolution is described by the vertical vorticity equations, $\omega_3 = -\nabla_H^2 \psi$

$$\nabla_H^2 \dot{\psi} + v_j \nabla_H^2 \psi_{,j} = \nu \nabla_H^2 \psi_{,jj} \, . \tag{27}$$

as can be verified by taking the curl. As the horizontal divergence vanishes, as $Fr_t^2 \to 0$ the equations are frame-invariant and the evolution is frame-indifferent. The ψ solution generates the velocities and the pressure. The temperature is found from the pressure through the hydrostatic balance; the energy equation is then a diagnostic relationship for w. It should be noted that the streamfunction has vertical structure and that the turbulence is frame-indifferent.

In the low Fr_t the waves and the stratified turbulence are describable by a Helmholtz decomposition for the velocity field is possible:

$$u_p = \;\; \epsilon_{pqk} \hat{\Omega}_k \psi_{,q} \; + \phi_{,p} \; - \phi_{,3} \, \delta_{p3} \; + \; w \delta_{p3} \tag{28}$$

The horizontal divergence of the field is associated with the wave field: as $u_{p,p} = 0$ thus $-\nabla_H^2 \phi = w_{,3}$. Such a velocity field decomposition seems to have first been used, in this context, by Riley, Metcalf and Weissman (1981).

5. Frame-invariance of the second-moment equations

A similar investigation of the frame invariance of the second-moment equations is now described. It will be shown (1) how the geostrophic balance manifests itself in the second moment equations, and (2) what constraints any representation for the rapid-pressure covariances are required to satisfy. It should be clear that a geostrophic turbulence is one whose horizontal divergence is small because its Rossby number is small reflecting a pressure-Coriolis balance $2\varepsilon_{ikp} u_p \Omega_p \simeq -p_{,i}$. However any flow with vanishing horizontal divergence (and arbitrary Rossby number) also satisfies the geostrophic constraint. Both have second-moment evolution equations that are frame-invariant.

The instantaneous fields are partitioned into mean and fluctuating field. The usual Reynolds decomposition, $u_i^* = U_i + u_i$ and $T^* = T + \theta$, is used. The second-order moment equations for an

incompressible turbulence, in the Boussinesq approximation, in a rotating coordinate system, are

$$\frac{D}{Dt} < u_i u_j > + 2\,\hat{\Omega}_k Ro^{-1}[\epsilon_{ikp} < u_p u_j > + \epsilon_{jkp} < u_p u_i >] = \; < \theta u_i > \beta_j + < \theta u_j > \beta_i$$
$$-[< u_j u_p > U_{i,p} + < u_i u_p > U_{j,p}] \qquad - < u_i u_j u_p >_{,p}$$
$$-[< p_{,j}\, u_i > + < p_{,i}\, u_j >] + Re^{-1} < u_i u_j >_{,pp} - \qquad 2Re^{-1} < u_{i,p}\, u_{j,p} > \tag{29}$$

$$\frac{D}{Dt} < \theta u_i > + 2\epsilon_{pik}\hat{\Omega}_k < \theta u_p > Ro^{-1} = \quad -[< \theta u_j > U_{i,j} + < u_i u_j > T_{,j}] + < \theta\theta > \beta_i$$
$$- < \theta u_i u_j >_{,j} - < p_{,i}\theta >$$
$$+ Re^{-1}(1 + Pr^{-1})(< \theta u_i >_{,jj} - 2 < \theta_{,j} u_{i,j} >). \tag{30}$$

The velocity has been normalized by a characteristic turbulence velocity u_c and the Rossby number is $Ro = u_c/\Omega R_c$ where R_c is a length scale and Ω the rotation rate of the frame of reference. The gravity and rotation vectors are aligned with the 3 axis. The concern is with the pressure-strain and pressure-temperature gradient correlations, $-[< p_{,j}\, u_i > + < p_{,i}\, u_j >] = -[< pu_i >_{,j} + < pu_j >_{,i}] + < ps_{ij} >$ and $< p_i\theta > = < p\theta >_{,i} - < p\theta_i >$. An equation for the pressure fluctuations comes from the divergence of the Navier-Stokes equations for the fluctuating velocity

$$u_{i,t} + u_j U_{i,j} + U_j u_{i,j} + u_j u_{i,j} - < u_i u_j >_{,j} + 2\epsilon_{ikp}\hat{\Omega}_k u_p Ro^{-1} = -p_{,i} + \theta\beta_i + Re^{-1} u_{i,jj} \tag{31}$$

which produces a Poisson equation for fluctuating pressure. The standard linear decomposition recognizes three terms

$$-p^r_{,ii} = \; 2[U_{i,p} + \epsilon_{pik}\hat{\Omega}_k Ro^{-1}] u_{p,i} \tag{32}$$
$$-p^s_{,ii} = \; u_{i,j}\, u_{j,i} - < u_{i,j}\, u_{j,i} > \tag{33}$$
$$p^b_{,ii} = \; \beta_i \theta_{,i} \tag{34}$$

where p^r, p^s, p^b are respectively the rapid-pressure, the slow or return to isotropy pressure, and the buoyancy-pressure. The effects of rotation are felt through the rapid-pressure, p^r. Solution of the Poisson equation for the rapid-pressure is by application of Green's theorem

$$\phi(\mathbf{x}) = -\,(4\pi)^{-1} \int \phi(\mathbf{x}')_{,jj} \frac{d\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}.$$

It is the moments of the solution that are required to close the second-order equations. For a homogeneous mean field a straightforward interchange of the order integration and averaging produces:

$$< p^r s_{ij} > = \; -2[U_{q,p} + \epsilon_{pqk}\hat{\Omega}_k Ro^{-1}][X_{ipqj} + X_{jpqi}] \tag{35}$$
$$< p^r \theta_{,i} > = \; -2[U_{q,p} + \epsilon_{pqk}\hat{\Omega}_k Ro^{-1}] X_{piq} \tag{36}$$

where $s_{ij} = \frac{1}{2}[u_{i,j} + u_{j,i}]$ and

$$X_{piq} = (4\pi)^{-1} \int < \theta(\mathbf{x}) u_p(\mathbf{x}') >_{,i'q'} \frac{d\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|} \tag{37}$$

$$X_{ipqj} = (4\pi)^{-1} \int < u_i(\mathbf{x}) u_p(\mathbf{x}') >_{,j'q'} \frac{d\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}. \tag{38}$$

By continuity $s_{jj} = 0$; this $< ps_{ij} >$ interchanges energy between the different components of the Reynolds stress but does not contribute to the kinetic energy of the turbulence. In drawing the mean velocity gradient outside the integral the assumption of quasi-homogeneity has been made: the length scale of the inhomogeneity of the mean field is large compared to the integral scale of the turbulence. The primary contribution to the integral will then come from regions within an integral length scale of the local position over which the velocity gradient is approximately constant.

Consider the portion of the rapid-pressure correlation associated with the rotation

$$\epsilon_{pqk}\hat{\Omega}_k Ro^{-1} X_{ipqj} = \frac{1}{4\pi Ro} \int \epsilon_{pqk}\hat{\Omega}_k < u_i(\mathbf{x}) u_p(\mathbf{x}') >_{,j'q'} \frac{d\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}. \tag{39}$$

A horizontally divergence free velocity field has, to within an arbitrary additive scalar function, the representation

$$u_p = \epsilon_{pqk}\hat{\Omega}_k \psi_{,q}\,, \qquad \psi_{,p} = \epsilon_{qpk}\hat{\Omega}_k u_q + \hat{\Omega}_p \hat{\Omega}_q \psi_{,q} \tag{40}$$

for $\psi = \psi\,(x, y, z)$. The flow along the rotation axis is specified by an additive scalar function. In the stably stratified case in which there is a horizontal divergence associated with potential field the analysis is the same: the potential makes no contribution to the integral. A proof of the frame-invariance of the second-moment equations when the velocity field has this representation is straightforward. Insert the expression for the velocity field into the integral and contract to produce, in the integrand, the Laplacian of the streamfunction, $\psi_{,p} = \epsilon_{qpk}\Omega_k u_q$ which reduces the volume integral of a two-point statistic to a one-point statistic which is identical with the Coriolis term,

$$\frac{1}{4\pi} \int \epsilon_{pqk}\hat{\Omega}_k < u_i(\mathbf{x}) u_p(\mathbf{x}') >_{,j'q'} \frac{d\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|} = < u_i \psi_{,j} > = \epsilon_{pjk}\hat{\Omega}_k < u_i u_p > . \tag{41}$$

This analysis and a similar one for the heat flux equations produce the geostrophic constraints

$$\epsilon_{pqn}\Omega_n X_{ipqj} = \epsilon_{qjn}\Omega_n < u_i u_q > \tag{42}$$

$$\epsilon_{pqn}\Omega_n X_{pqi} = \epsilon_{qin}\Omega_n < u_q \theta > . \tag{43}$$

These are constraints reflecting the geostrophic balance of the second-moments of the fluctuating field. The indicated contraction of any model for the rapid-pressure strain covariance, X_{ipqj} and

X_{pqi} is required to satisfy. From another point of view: when the fluctuating flow is generated by a streamfunction the rapid-pressure covariance representation equals the Coriolis terms and the exact equations (and therefore their models) are frame-indifferent. This is not an invocation of the principle of material-frame-indifference, Lumley (1970), used in rational mechanics to construct a constitutive relation. It is a statement, derivable from first principles, that any constitutive relation for the rapid-pressure strain must satisfy if the evolution equations are frame-indifferent *in the limit* of a two-dimensional or a two-componential turbulence.

6. Additional mathematical properties of the rapid-pressure covariance

There are additional mathematical properties that the rapid-pressure covariance integrals must satisfy. The fourth- and third-order tensors have the following symmetries: $X_{ijkl} = X_{ijlk}$, $X_{ijkl} = X_{jikl}$, $X_{ijk} = X_{ikj}$. For an arbitrary three-dimensional turbulence the tensor polynomials must also satisfy the constraints of normalization and continuity:

$$\begin{aligned} X_{ijkk} &= \ < u_i u_j >, \ X_{ikk} = \ < u_i \theta >, \\ X_{ijjk} &= \qquad 0, \ X_{iji} = \qquad 0. \end{aligned} \tag{44}$$

Note that a contraction of the integral of a two-point statistic is a local one-point statistic. For a purely isotropic turbulence the tensors have the values

$$X_{ijkl} = \frac{4}{15}\, k\, [\delta_{ij}\delta_{kl} - \frac{1}{4}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})] \tag{45}$$

$$X_{ijk} = 0 \tag{46}$$

The tensor $< u_i u_j >$ has positive eigenvalues. This reflects the fact that the energy of the turbulence is always positive and that the magnitude of the correlation coefficients between the various components of the tensors be bounded by one. These facts lead to "realizability" constraints that specify the behavior of the correlations as an eigenvalue of the Reynolds stress approaches zero. The relevant portion of the Reynolds stress transport equations, in principal axes, requires that

$$D/Dt \ < u_\alpha u_\alpha > \sim \ [U_{p,i} + \epsilon_{ipk}\Omega_k Ro^{-1}] X_{i\alpha p\alpha} \rightarrow 0 \quad \text{as} \qquad < u_\alpha u_\alpha > \rightarrow 0 \tag{47}$$

in order to satisfy realizability. The rate of change, due to the rapid-pressure correlation, of the eigenvalue $< u_\alpha u_\alpha >$ is required to vanish as the limit state is approached. This insures that the rapid-pressure correlation model does not cause the solution to go into the unrealizable region in which $< u_\alpha u_\alpha >$ is negative. This is extremely dangerous from a computational viewpoint, Ristorcelli *et al.* (1995). The realizability limit is rephrased in terms of the determinant of the Reynolds stress: $F = (R^3_{jj} - 3R_{jj}R^2_{jj} + 2R^3_{jj})/6$ where $R_{ij} =< u_i u_j > / < u_p u_p >$ which can be written in terms of the invariants of the anisotropy tensor as $F = 1 + 9II + 27III$ where

$II = -\frac{1}{2}b_{ij}b_{ij} = -\frac{1}{2} < b^2 >$, $III = \frac{1}{3}b_{ip}b_{pj}b_{ji} = \frac{1}{3} < b^3 >$. The determinant F varies between zero and one; $F = 1$ corresponds to an isotropic turbulence and $F = 0$ corresponds to the realizable limit in which an eigenvalue is zero.

The quantities X_{piq} and X_{ipqj} cannot be modeled independently; and the heat fluxes and the Reynolds stresses are linked through Cauchy-Schwarz type inequalities, $D_{ij} = < \theta\theta >< u_i u_j > - < \theta u_i >< \theta u_j > \geq 0$. D_{ij} has positive semi-definite eigenvalues. Similar reasoning produces the "joint-realizability" constraint

$$D/Dt\, D_{\alpha\alpha} \sim [U_{p,i} + \epsilon_{ipk}\Omega_k Ro^{-1}][< \theta\theta > X_{i\alpha p\alpha} - < \theta u_\alpha > X_{i\alpha p}] \to 0 \quad \text{as} \quad D_{\alpha\alpha} \to 0 \quad (48)$$

which couples the rapid-pressure correlations appearing in the heat flux and the Reynolds stress equations. A similar determinant function F_d is defined with the normalized D_{ij}, for which $0 \leq F_d \leq 1$. Joint-realizability reflects the requirement that the magnitude of the correlation coefficients be bounded by one: the Reynolds stress and the heat-flux take on values "jointly" such that the time rate of change of $D_{\alpha\alpha}$ goes to zero as $D_{\alpha\alpha}$ goes to zero. The mathematical properties, continuity, symmetry, normalization, isotropy, geostrophy, and realizability of the unknown rapid-pressure covariances have now been given. The possibility of a constitutive relation is now addressed.

7. A constitutive relation for the rapid-pressure covariance

A closure for unknown rapid-strain covariances,

$$X_{piq} = \frac{1}{4\pi} \int < \theta(\mathbf{x}) u_p(\mathbf{x}') >_{,i'q'} \frac{d\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|} \quad (49)$$

$$X_{ipqj} = \frac{1}{4\pi} \int < u_i(\mathbf{x}) u_p(\mathbf{x}') >_{,j'q'} \frac{d\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|} \quad (50)$$

is now sought. The two-point velocity and velocity-temperature covariances satisfy complex evolution equations and the quantity desired is the integral of their double divergence. There is little that can be done at this point; not much of any simplicity will result from investigations of the two-point evolution equations. A hypothesis which cannot be true with full generality but which may be rigorously correct in special circumstances is adopted.

The point of view of rational mechanics, Lumley (1970), is adopted: if a constitutive relation is to be determined phenomenologically it must satisfy certain general properties that the original unknown quantity satisfies. These properties, tensor invariance, symmetry, continuity, normalization, geostrophy, isotropy and realizability, must therefore be satisfied by any constitutive relation for the rapid-pressure covariances. In this way a general structure for the phenomenological relation is obtained. One might then expect the closure to be able to predict the pressure-strain for a class of flows from another of the same class. The subject of this article is closure in terms of other single-point quantities carried in such a closure. The properties of continuity, geostrophy, and isotropy

which relate the unknown integrals, X_{piq}, X_{ipqj}, to single-point quantities suggest such a closure, may be tenable.

A closure of the form $X_{piq} = X_{piq}(< u_i u_j >, < u_i \theta >)$ and $X_{ipqj} = X_{ipqj}(< u_i u_j >)$ is sought. Parameterizing the integrals of a two-point covariance in terms of the Reynolds stress and the heat flux is a substantial simplification requiring consideration. It is likely only to be adequate for certain classes of turbulent flows. Lumley (1970) has discussed the conditions under which such a constitutive relation is possible. Lumley's (1970) conclusions are that such a procedure is suitable for slowly evolving flows,

$$\frac{\overset{\circ}{S}}{S}\frac{k}{\varepsilon} < 1, \quad or \quad \frac{\overset{\circ}{\ell}}{\ell}\frac{k}{P_k} < 1 \tag{51}$$

suitably removed from the initial conditions

$$\Delta t \; \frac{P_k}{k} > 1 \tag{52}$$

so that nonlinear effects and production mechanisms that generate turbulence have been in operation long enough to decorrelate the present state from the initial state. Here Δt represents time past initiation of the flow, the overdot represents a Lagrangian derivative, S represents some norm of the strain S_{ij}, P_k is the production rate of k and ℓ is an integral scale of the turbulence. The overdot represents the Lagrangian derivative. For a turbulence with short term memory and limited awareness Lumley (1967) has carried out an expansion procedure indicating how truncation errors might scale.

There is another point of view that is useful to consider. A brief summary is given - these points have been covered clearly in Lumley (1970, 1978), and Lumley and Khajeh-Nouri (1974). Consider the high Reynolds number homogeneous form of the Reynolds equations

$$\begin{aligned}\frac{D}{Dt} < u_i u_j >= \; & - [< u_j u_p > U_{i,p} + < u_i u_p > U_{j,p}] - \tfrac{2}{3}\varepsilon\delta_{ij} \\ & -2[U_{q,p} + \epsilon_{pqk}\Omega_k Ro^{-1}][X_{ipqj} + X_{jpqi}]\end{aligned} \tag{53}$$

as an equation for X_{ijkl}; X_{ijkl} can then be written as functionals of the Reynolds stresses, the dissipation, the heat fluxes throughout the field and over previous time; $X_{ijkl} = X_{ijkl}\{\nabla\mathbf{U}, < \mathbf{uu} >, \varepsilon\}$, $X_{ijk} = X_{ijk}\{\nabla\mathbf{U}, < \mathbf{uu} >, < \mathbf{u}\theta >, \varepsilon_\theta\}$. Expanding the functional in a Taylor series about the present state and keeping only the lowest order terms is suitable for a slowly evolving turbulence. Retaining higher order terms of the functional Taylor series expansion will involve spatial and temporal derivatives of b_{ij} and substantially complicate the problem. It is expected that, *for a slowly evolving turbulence*, retention of only the first order term captures enough of the physics

to allow predictions suitable for calculations provided that any model for X_{ijkl} and X_{pkj} have the proper mathematical properties. This is a brief recapitulation of the arguments that lead to the possibility of closures using constitutive relations. It is clear that the theoretical development underlying this procedure is not applicable to rapidly evolving flows or to flows whose anisotropy is not parameterizable by $< u_i u_j >$ and $< u_i \theta >$.

The most general forms of a parameterizations for the fourth-order tensor appearing in the Reynolds stresses will be

$$\begin{aligned}
X_{ijkl}/2k = \; & A_1 \delta_{ij} \delta_{kl} + A_2 [\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}] \\
+ \; & A_3 \delta_{ij} b_{kl} + A_4 b_{ij} \delta_{kl} + A_5 [b_{ik} \delta_{jl} + b_{il} \delta_{jk} + \delta_{ik} b_{jl} + \delta_{il} b_{jk}] \\
+ \; & A_6 \delta_{ij} b^2_{kl} + A_7 b^2_{ij} \delta_{kl} + A_8 [b^2_{ik} \delta_{jl} + b^2_{il} \delta_{jk} + \delta_{ik} b^2_{jl} + \delta_{il} b^2_{jk}] \\
+ \; & A_9 b_{ij} b_{kl} + A_{10} [b_{ik} b_{jl} + b_{il} b_{jk}] \\
+ \; & A_{11} b_{ij} b^2_{kl} + A_{12} b^2_{ij} b_{kl} + A_{13} [b^2_{ik} b_{jl} + b^2_{il} b_{jk} + b_{ik} b^2_{jl} + b_{il} b^2_{jk}] \\
+ \; & A_{14} b^2_{ij} b^2_{kl} + A_{15} [b^2_{ik} b^2_{jl} + b^2_{il} b^2_{jk}].
\end{aligned}$$

The six Reynolds stresses have been replaced with the six quantities b_{ij} and $k = \frac{1}{2} < u_p u_p >$ where $b_{ij} =< u_i u_j > /2k - \frac{1}{3} \delta_{ij}$ is the anisotropy tensor. Following Pope's (1983) linearity principle only terms linear in the heat-flux are kept in the tensor polynomial for the representation rapid-presure in the heat flux equations:

$$\begin{aligned}
X_{pkj} = \; & D_1 < \theta u_p > \delta_{kj} + D_2 [< \theta u_k > \delta_{pj} + < \theta u_j > \delta_{pk}] \\
+ \; & D_3 < \theta u_p > b_{kj} + D_4 [< \theta u_k > b_{pj} + < \theta u_j > b_{pk}] \\
+ \; & D_5 < \theta u_p > b^2_{kj} + D_6 [< \theta u_k > b^2_{pj} + < \theta u_j > b^2_{pk}] \\
\\
+ \; & [D_7 b_{qp} \delta_{kj} + D_8 (b_{qk} \delta_{pj} + b_{qj} \delta_{pk})] < \theta u_q > \\
+ \; & [D_9 b_{qp} b_{kj} + D_{10} (b_{qk} b_{pj} + b_{qj} b_{pk})] < \theta u_q > \\
+ \; & [D_{11} b_{qp} b^2_{kj} + D_{12} (b_{qk} b^2_{pj} + b_{qj} b^2_{pk})] < \theta u_q > \\
\\
+ \; & [D_{13} b^2_{qp} \delta_{kj} + D_{14} (b^2_{qk} \delta_{pj} + b^2_{qj} \delta_{pk})] < \theta u_q > \\
+ \; & [D_{15} b^2_{qp} b_{kj} + D_{16} (b^2_{qk} b_{pj} + b^2_{qj} b_{pk})] < \theta u_q > \\
+ \; & [D_{17} b^2_{qp} b^2_{kj} + D_{18} (b^2_{qk} b^2_{pj} + b^2_{qj} b^2_{pk})] < \theta u_q > .
\end{aligned}$$

The A_i and D_i are scalar *functions* of the invariants of b_{ij} and $< \theta u_i >$. They are functions of the state of the flow as characterized by the invariants: they are not constants, Rivlin (1955), Spencer (1971). As complex as these expressions appear one should keep in mind that they result from nothing more complex, conceptually, than the Buckingham *PI* theorem; tensor quantities require invariance under a larger group of transformations.

This methodology can be contrasted to the usual procedure. The usual modeling procedure is to choose these functions to be constants and then use the empirical curve fitting procedure with modeled partial differential equations to match predictions for the Reynolds stresses in a number of

flows. Sometimes these constitutive relations for the rapid-pressure strain also require expressions linear in the b_{ij}; such a requirement does not satisfy many of the required mathematical properties.

A few caveats appear necessary. These sorts of constitutive arguments, that allow $X_{piq} = X_{piq}(< u_i u_j >, < u_i \theta >)$ and $X_{ipqj} = X_{ipqj}(< u_i u_j >)$, are not applicable to rapidly distorted flows; these are essentially linear problems in which the dependence on the initial conditions has not been decorrelated by the nonlinear processes of turbulence. The assumption of a slowly evolving turbulence underlying the constitutive relation procedure is violated. The shortcomings of this sort of methodology for rapidly distorted problems are known, Reynolds (1994), and the shortcomings of the linear rapid distortion methods for fully nonlinear turbulence problem are also well known. Our interest here is in geophysical flows; the atmosphere and the ocean have been integrating for several million years, Margolin (1996), and for these geophysical flows, as is well known for fully developed turbulence in engineering flows, terms quadratic in the fluctuations are far more important than those linear, Smith (1996).

Inserting the tensor polynomial expressions into the continuity, normalization, geostrophy and realizability constraints produces a set of algebraic equations. The solution for the algebraic equations yields the following expressions for the scalar functions:

$$\begin{aligned}
A_1 =& \ (111II + 73)/27II_d - F(420II + 239)/135II_d \\
A_2 =& \ -(69II + 32)/27II_d + F(420II + 257)/270II_d \\
A_3 =& \ (3II + 4)/3II_d - F(11/10)/(1 + 3II) \\
A_4 =& \ (15II + 11)/3II_d - F(4/10)/(1 + 3II) \\
A_5 =& \ -3(1 + 3II)/3II_d + F(3/10)/(1 + 3II) \\
A_6 =& \ -(102II + 61)/3II_d \\
A_7 =& \ -2(33II + 20)/3II_d - F(6/10)/(1 + 3II) \\
A_8 =& \ (42II + 23)/3II_d \\
A_9 =& \ -(57II + 28)/3II_d - F(3/10)/(1 + 3II) \\
A_{10} =& \ (15II + 14)/3II_d + F(9/10)/(1 + 3II) \\
A_{11} =& \ -(102II + 61)/II_d \\
A_{12} =& \ -2(33II + 20)/II_d \\
A_{13} =& \ (42II + 23)/II_d
\end{aligned}$$

where $A_{14} = 0$ and $A_{15} = 0$ and $II_d = (1 + 3II)(7 + 12II)$, $F = 1 + 27III + 9II$, where $II =$

$-1/2b_{ij}b_{ij}$ and $III = 1/3b_{ip}b_{pj}b_{ji}$. In the third-order tensor polynomial the scalar functions are

$$
\begin{aligned}
D_1 =& \; -(312II^2 + 149II - 21)/5II_d - F(1/5)/(1+3II) \\
D_2 =& \; (48II^2 + II - 14)/5II_d + F(3/10)/(1+3II) \\
D_3 =& \; -(324II^2 + 222II + 17)/II_d - F(3/10)/(1+3II) \\
D_4 =& \; -3/(7+12II) + F(9/10)/(1+3II) \\
D_5 =& \; -(102II + 61)/II_d \\
D_6 =& \; (42II + 23)/II_d \\
D_7 =& \; -2(3II + 4)/5II_d - F(3/5)/(1+3II) \\
D_8 =& \; 27(2II + 1)/5II_d \\
D_9 =& \; (42II + 23)/II_d \\
D_{10} =& \; (42II + 23)/II_d \\
D_{13} =& \; -8(39II + 22)/5II_d \\
D_{14} =& \; 2(24II + 17)/5II_d \\
D_{15} =& \; -27/(1+3II).
\end{aligned}
$$

Note that $D_{11} = D_{12} = D_{16} = D_{17} = D_{18} = 0$ as well as $A_{14} = A_{15} = 0$ have been set to zero as they are not necessary to satisfy the mathematical constraints; there are some free parameters left in this closure. Recourse to experimental data has not yet been made. What is now required is a flow that has attained a structural equilibrium, $\frac{D}{Dt}b_{ij} = 0$; the fixed points of the modeled evolution equations for b_{ij} are then required to match the phenomenological data. This provides six additional algebraic constraints to set the free parameters. This is the requirement that the constitutive relation be asymptotically consistent with a known equilibrium state of a particular turbulent flow. This procedure was introduced by Speziale, Sarkar and Gatski (1990). It is executed for the present rapid-pressure strain closure for a homogeneous simple shear in Ristorcelli *et al.* (1995).

8. Additional commentary

In the Fourier space the rapid-pressure covariance can be written as an integral of the energy spectrum over all the scales of the motion: from the production scales at $\kappa\ell \sim 1$ and larger to the dissipation scales $\kappa\eta \sim 1$. The major contribution to the integral will be from large scales of the motion, $\kappa\ell \sim 1$. In a high Reynolds number turbulence with a $\kappa^{-5/3}$ inertial subrange in which there is enough of a separation of scales for a second-order simulation to be useful approximation, say at least $Re_\ell \sim 10^4$, the ratio between the dissipative and the energy containing length scales is $\eta/\ell \sim Re_\ell^{-3/4} \sim 1000$ and the flow scales range over $0 < \kappa\ell < 1000$. Approximately 85% of the energy of the motion is contained in the first decade $\kappa\ell < 10$: the major contribution to the rapid pressure integral is from the scales of the motion greater than one tenth of the production scales. If only the largest 1% of the flow scales, *i.e.* from $0 < \kappa\ell < 10$, begins to lose an eigenvalue of the Reynolds stress tensor, the rapid-pressure will begin to approach the frame-indifferent limit. The rapid-pressure covariance becomes *asymptotically close to the frame-invariant limit with only*

a relatively small portion of the scales of the flow becoming horizontally divergence free.

Consideration is now given to additional limitations to the class of flows for which the constitutive relation posited is *not* appropriate. The anisotropy of the Reynolds stress is not the sole measure of the anisotropy of a turbulence field. Consider homogeneous rotating isotropic turbulence as seen in the DNS of Mansour, Cambon and Speziale (1992). As assessed by Reynolds stresses it remains isotropic yet the two-point statistics are highly anisotropic indicating a larger correlation length scale along the axis of rotation. Reynolds (1989), Kassinos and Reynolds (1994), Reynolds and Kassinos (1994) have realized the inadequacy of single-point closures to account for effects of this form of anisotropy expected to be important in nonequilibrium situations. Within the context of a single-point closure the resolution of this problem can only be accomplished by carrying an additional quantity to represent this missing information. Reynolds (1989) uses single point quantity that he calls the structure dimensionality tensor that appears to be a useful proxy for the information contained in two-point quantities. The possibility of the method to account for two-point information using a single-point quantity is an interesting idea worth following.

The rapid-pressure strain covariance which is an integral over the two-point covariance is very much affected by the two point anisotropy. The implicit assumption in single-point developments with closures parameterized by Reynolds stresses is the implicit assumption of a single characteristic length scale. This is a reasonable assumption for slowly evolving fields in which nonlinear effects strongly couple the different directions. In geophysical flows in which body forces reduce the nonlinear strain-vorticity interaction mechanism, Tennekes and Lumley (1972), the two-point behavior is substantially different. The present version of single-point formalism can nonetheless be applied to these flows; the physics that produces the anisotropy of the two-point statistics also produces the anisotropy in the Reynolds stress. Rotating geophysical flows such as the planetary boundary layers occur adjacent to solid boundaries and Taylor-Proudman requires $u_3 \rightarrow 0$. Thus in either rapidly rotating or stably stratified flows the turbulence becomes a horizontal turbulence, $< u_3 u_3 > \rightarrow 0$, for which a single-point parameterization using $< u_i u_j >$ and $< u_i \theta >$ is possible. This will also be the case for strongly vortical flows in which the vorticity of the fluctuating field is aligned with the mean rotation.

In the DNS of homogeneous rotating isotropic turbulence a decrease in the nonlinear cascade rate is noticed. This has been explained as an interference of the inertial wave field associated with rotation with the phase coherence necessary for the cascade of energy to the smaller scales of the motion. Here it has been shown that as the flow becomes horizontally divergence free the transfer between the vertical and the other components of vorticity is reduced. This is an important

component of the cascade process as Tennekes and Lumley (1972) have shown. Whatever the mechanism the reduction of the cascade mechanism will make the current parameterizations for the dissipation equation invalid. There are some limits on the rotation rate that can be set in order for the current parameterization of the cascade to be adequate. Consider the Rossby number defined as a ratio of the vorticity of the production scales of the motion to the background vorticity $Ro_t = (q^2/3)^{1/2}/2\Omega\ell = 3\varepsilon/2\Omega q^2$ using $\varepsilon = (q^2/3)^{1/2}/\ell$. A spectral Rossby number can also be defined as $Ro_t(\kappa) = u(\kappa)/2\Omega\ell(\kappa) = (\kappa E(\kappa))^{1/2}/2\Omega(2\pi/\kappa)$ which using the inertial range scaling $E(\kappa) = \alpha\varepsilon^{2/3}\kappa^{-5/3}$ and $\varepsilon = (q^2/3)^{1/2}/\ell$ becomes $Ro_t(\kappa) \sim 0.2\ (\kappa\ell)^{2/3}Ro_t$. The effects of rotation decrease as the wave number increases. For the inertial oscillations associated with the rotation not to interfere with the cascade mechanism $Ro_t(\kappa) > 1$ for $\kappa\ell \sim 2$ is required. Thus for $Ro_t > 3$ the usual parameterization of the spectral cascade rate, ε, in terms of the energy containing scales of the motion is appropriate. For $Ro_t < 3$ the current dissipation equation begins to require modification. How the dissipation equation is to rationally account for the effects of rotation on the cascade is an unresolved issue. It is, however, clear that the assumption of the small scale equilibrium with the large scales of the motion is valid in most high Reynolds number rotating flows of interest for $Ro_t > 1$: the Rossby number of the dissipation scales of the motion is $Ro_\varepsilon = Re_\ell^{1/2}Ro_t$ and therefore the dynamics of the small scale motions will be set by the large scales. At this point primarily phenomenological theories have been advanced, Zhou (1995), Mahalov and Zhou (1996). There is the rigorous analytical work of Babin, Mahalov and Nicolaenko (1996) on rotating turbulence that may well produce theoretical results that might bear on this issue in a rational way.

9. Summary and Conclusions

Three experiments - the rotating tank, the rotating heated annulus and the stratified grid turbulence - have been described. These cases cover three very basic classes of flows: rotating turbulence (in a finite domain), baroclinic turbulence (rotation with horizontal temperature gradient) and stably stratified turbulence. In the limit of small Rossby number or small Froude number these flows exhibit equilibrium states that are horizontally nondivergent and therefore frame-indifferent. If the second-moment equations are to be frame-invariant in the horizontally divergence free limit then rapid pressure-strain covariance must satisfy the following constraint:

$$\epsilon_{pqn}\Omega_n X_{ipqj} = \epsilon_{qjn}\Omega_n < u_i u_q > \tag{54}$$

$$\epsilon_{pqn}\Omega_n X_{pqi} = \epsilon_{qin}\Omega_n < u_q\theta > . \tag{55}$$

This equality, which assures the frame-invariance of the second-order equations, is called the geostrophic constraint as it reflects the small Rossby number geostrophic balance of the momentum equations, $2\varepsilon_{ikp}u_p\Omega_p \simeq -p_{,i}$. The geostrophic constraint must, however, be satisfied by any flow

with a vanishing horizontal divergence for *arbitrary* Rossby number; this is a manifestation of the fact that, in the absence of vortex stretching along the axis of rotation, that the evolution equations are frame-invariant. This is not an application of the material-frame-indifference principle of rational mechanics, Lumley (1970), it is a rigorous consequence of the Navier-Stokes equations. The evolution equations are quite dependent on rotation as long as the flow is not horizontally divergence free. The principle of frame-indifference in the two-dimensional limit, Speziale (1981,1985,1989), is a special case of this invariance relevant to a purely mechanical turbulence. Here the flow may have a vertical structure as might occur in stratified or baroclinic situations.

A horizontally nondivergent state has been shown to be a stationary flow achieved by turbulence under the influence of rapid rotation or strong stratification. The invariant basis representation methods have been used to construct a constitutive relation for the rapid-pressure strain covariance that, in the limit of a horizontally divergent free field, produces the proper frame-invariance. The representation constructed for the rapid-pressure strain covariance has variable coefficients. The coefficients are functions of the state of the turbulence and *are valid for all states of the turbulence* - they are not, as is typically the case, fixed to constant values obtained from empirical calibration, Ristorcelli *et al.* (1995). The constitutive relation for the rapid-pressure-strain covariance has been achieved for the class of horizontally nondivergent flows that are accompanied by the suppression of the component of the energy of the turbulence along the axis of rotation or stratification. This is a limited class of flows and represents the most that can be done with current single-point second-order turbulence closures without carrying additional equations for additional quantities. This limited class includes the horizontal turbulence case that occurs in most geophysical situations. These flows occur in such situations as the planetary boundary layers in which the flow transitions from a three-dimensional shear or convection driven mixed layer flow to an outer layer in geostrophic equilibrium. It also includes the strongly vortical flows in which the vorticity of the turbulence is aligned with that of the mean rotation.

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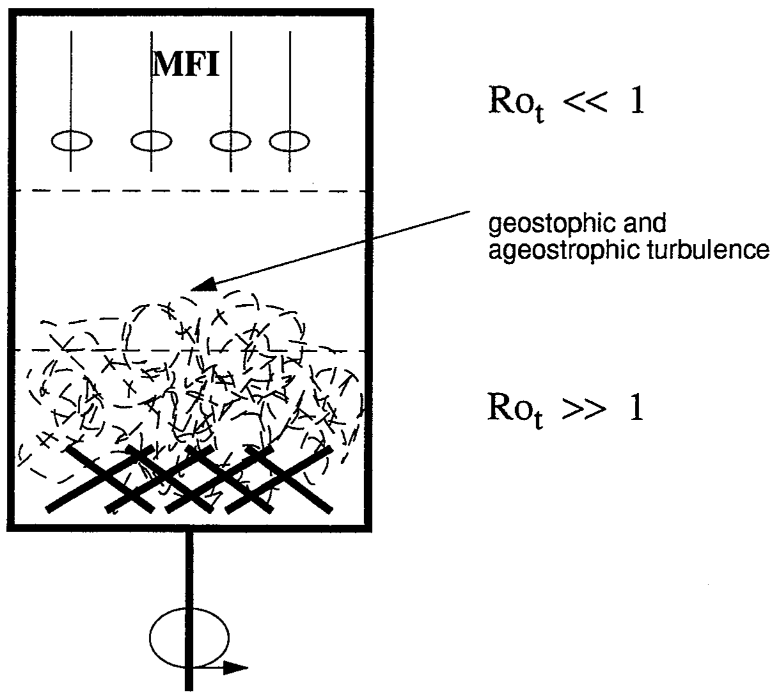

Figure 1. The oscillating grid in a rotating tank.

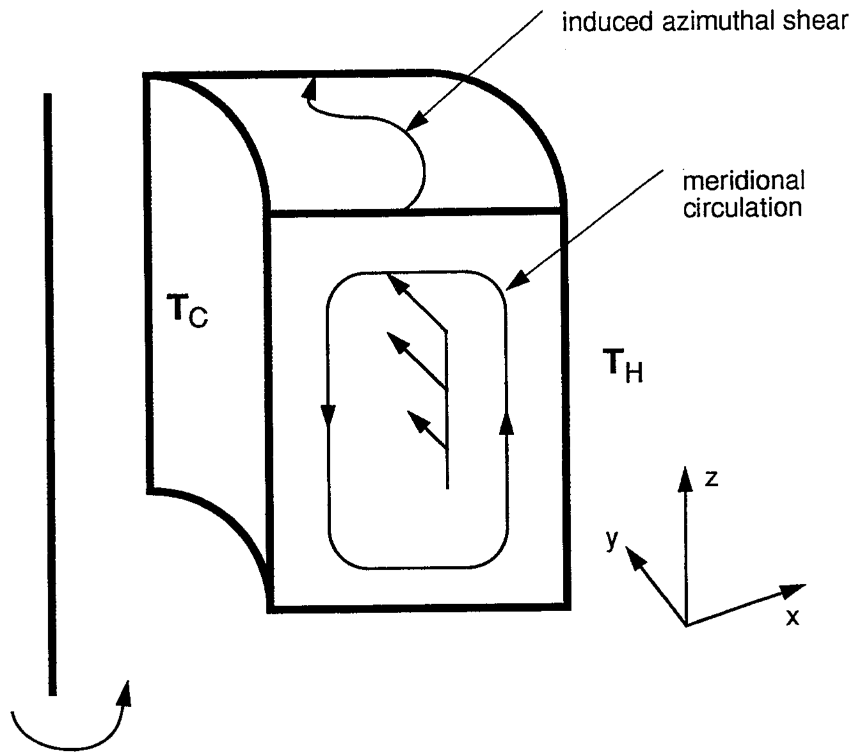

Figure 2. The differentially heated rotating annulus.

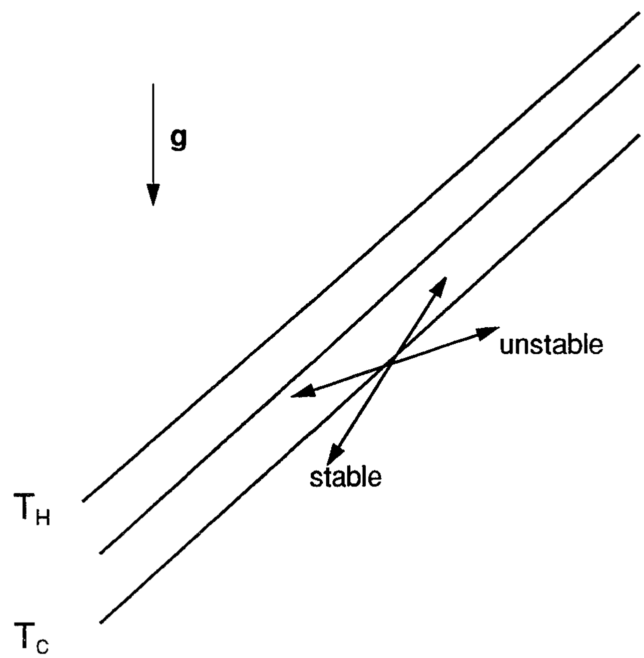

Figure 3. Schematic for the baroclinic instability.

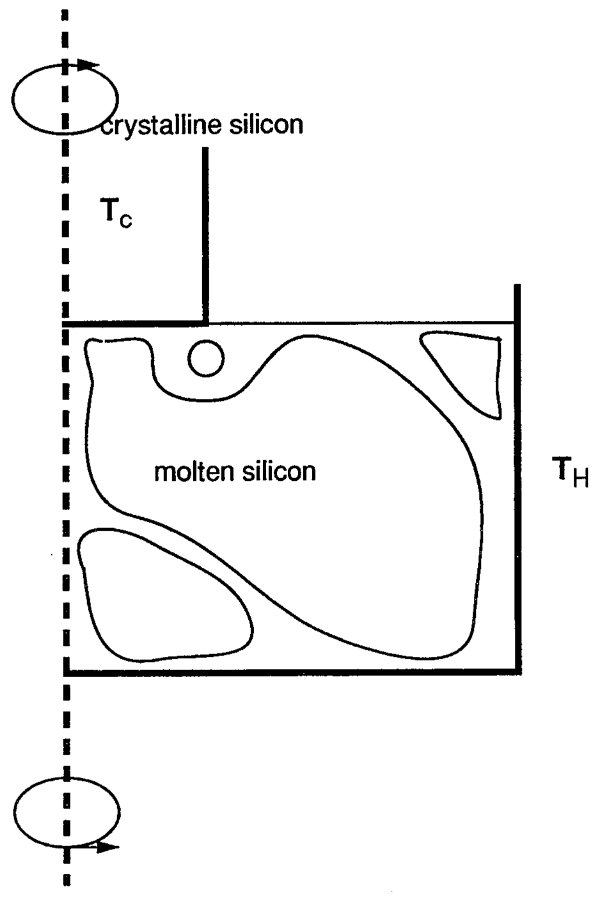

Figure 4. Czochralski crystal growth apparatus.

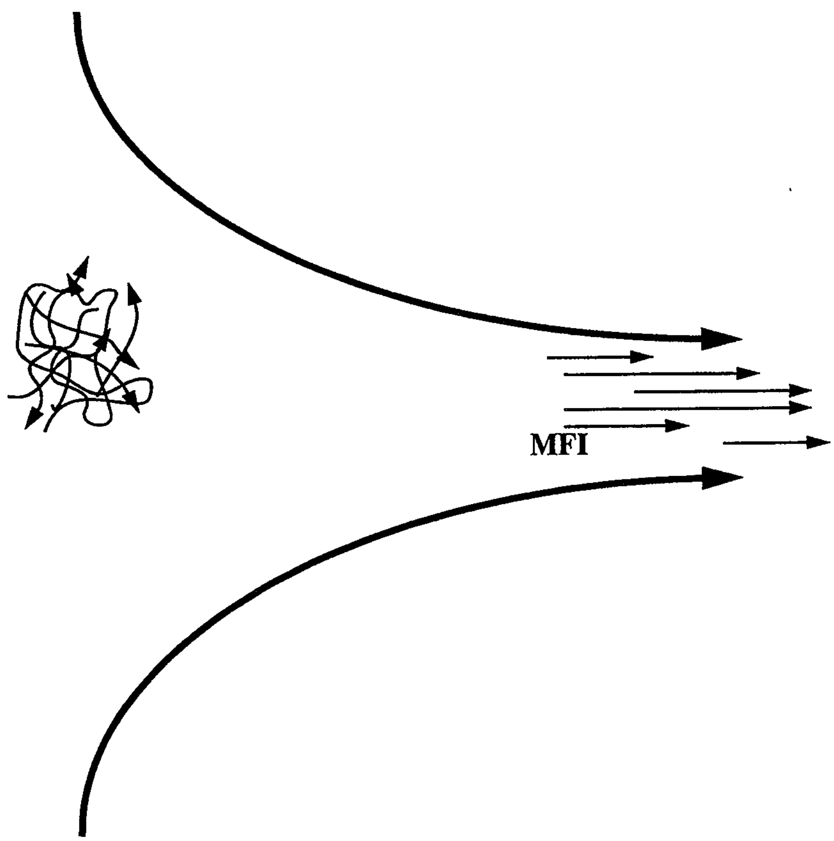

Figure 5. Strain aligning vorticity in axial direction.

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13. ABSTRACT *(Maximum 200 words)*

In rapidly rotating turbulent flows the largest scales of the motion are in approximate geostrophic balance. Single-point turbulence closures, in general, cannot attain a geostrophic balance. This article addresses and resolves the possibility of constitutive relation procedures for single-point second order closures for a specific class of rotating or stratified flows. Physical situations in which the geostrophic balance is attained are described. Closely related issues of frame-indifference, horizontal nondivergence, Taylor-Proudman theorem and two-dimensionality are, in the context of both the instantaneous and averaged equations, discussed. It is shown, in the absence of vortex stretching along the axis of rotation, that turbulence is frame-indifferent. A derivation and discussion of a geostrophic constraint which the prognostic equations for second-order statistics must satisfy for turbulence approaching a frame-indifferent limit is given. These flow situations, which include rotating and nonrotating stratified flows, are slowly evolving flows in which the constitutive relation procedures are useful. A nonlinear non-constant coefficient representation for the rapid-pressure strain covariance appearing in the Reynolds stress and heat flux equations consistent with the geostrophic balance is described. The rapid-pressure strain model coefficients are not constants determined by numerical optimization but are functions of the state of the turbulence as parameterized by the Reynolds stresses and the turbulent heat fluxes. The functions are valid for all states of the turbulence attaining their limiting values only when a limit state is achieved. These issues are relevant to strongly vortical flows as well as flows such as the planetary boundary layers, in which there is a transition from a three-dimensional shear driven turbulence to a geostrophic or horizontal turbulence.

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